Guaranteed interval integration for large initial boxes

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Content

Robotics context

- facing trajectory and path planning problems
- controller validation
- reaction to the environment in simulation

$\Rightarrow$ Need for a guaranteed integration tool, possibly lightweight

Current tools

In the domain of interval analysis tools have already been developed to perform guaranteed integration such as CAPD [3] or DynIbex [2]. These tools are based on conventional integration schemes (Runge-Kutta ...) to perform the calculation of the trajectory step by step.

Computer Assisted Proofs in Dynamics group

$\dot{\mathbf{y}} + \text{Ibex} = \mathbf{DynIbex}$

What is a Lie Group ? I

Definition

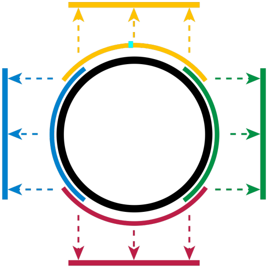
A **Lie Group** is a *smooth differentiable manifold* [1]

What is a Lie Group ? II

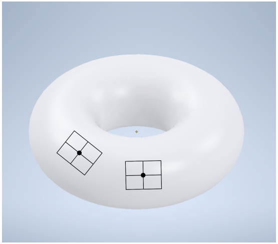
Examples of Lie groups:

Figure 1: Examples of lie groups

Why are they Lie Groups:



(a) unit circle



(b) the torus

Figure 2: Examples of lie groups 2

What is a Lie Group ? IV

Figure 3: Torus and one robot position

What is a Lie Group ? V

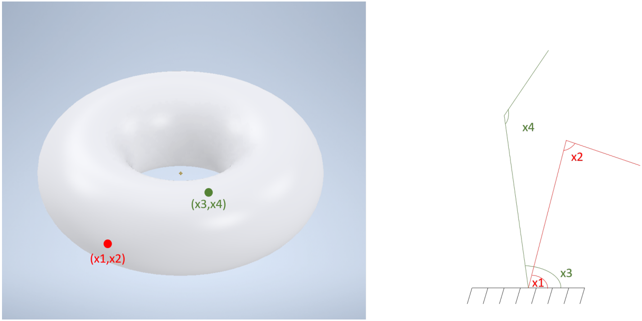


Figure 4: Torus and two robot positions

What is a Lie Group ? VI

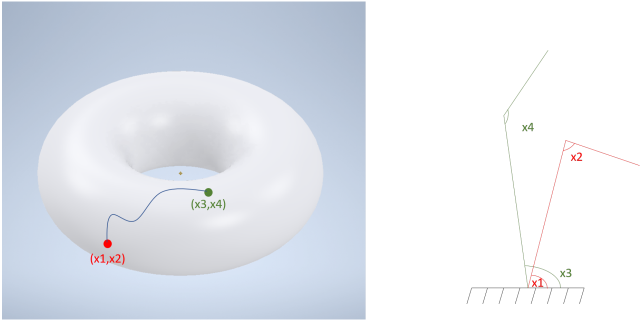


Figure 5: Trajectory on a Lie group

The transformations from one element to another are $\mathcal{C}^\infty$ i.e. the manifold is smooth

Action I

Definition

Given a state equation $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ where $\mathbf{x} \in \mathbb{R}^n$ and $\mathbf{g} : \mathbb{R}^n \to \mathbb{R}^n$ is a diffeomorphism, the *action* (noted $\bullet$) by $\mathbf{g}$ on $\mathbf{f}$ is defined by

$$\mathbf{g} \bullet \mathbf{f} = \left(\frac{d\mathbf{g}}{d\mathbf{x}} \circ \mathbf{g}^{-1} \right) * \left(\mathbf{f} \circ \mathbf{g}^{-1} \right) \quad (1)$$

Action II

Let's take the following system: $\begin{cases} \dot{x}_1 = -x_1^3 - x_1 x_2^2 + x_1 - x_2 \\ \dot{x}_2 = -x_2^3 - x_1^2 x_2 + x_1 + x_2 \end{cases}$

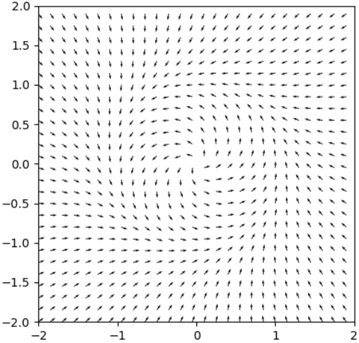


Figure 6: Vector Field associated to the system

Action III

Let us apply an action **g** on **f**, for example a symmetry around the vertical axis. The matrix associated to **g** is $\mathbf{G} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

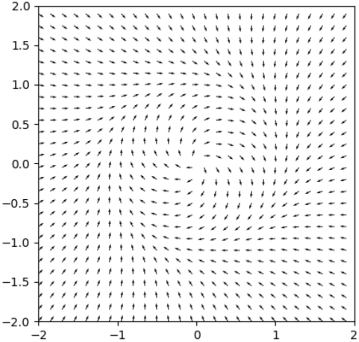


Figure 7: $\mathbf{g} \bullet \mathbf{f}$

Action IV

Another transformation can be reshaping the circle into an ellipse. It is done with $\mathbf{h}$ and its associated matrix is $\mathbf{H} = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$

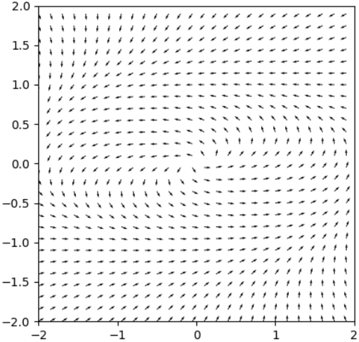


Figure 8: $\mathbf{h} \bullet \mathbf{f}$

Action V

It is possible to compose actions to create a third one

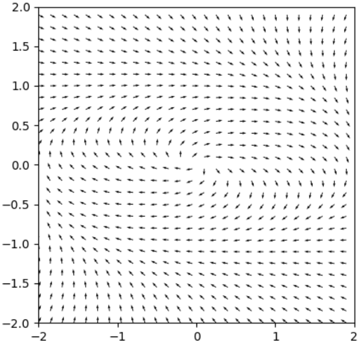


Figure 9: $(\mathbf{g} \circ \mathbf{h}) \bullet \mathbf{f}$

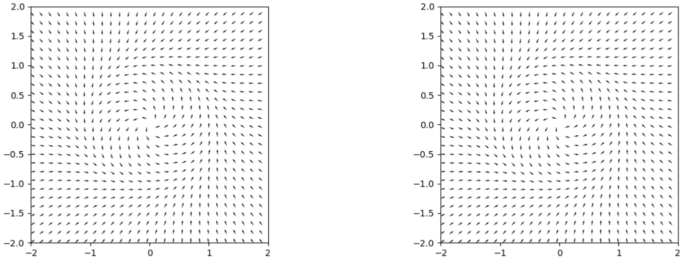
Stabilisers I

Definition

A transformation $\mathbf{g}$ is a *stabiliser* of $\mathbf{f}$ if $\mathbf{g} \bullet \mathbf{f} = \mathbf{f}$, i.e., if it satisfies the partial differential equation

$$\left(\frac{d\mathbf{g}}{d\mathbf{x}} \circ \mathbf{g}^{-1} \right) * \left(\mathbf{f} \circ \mathbf{g}^{-1} \right) = \mathbf{f} \tag{2}$$

Example of stabiliser: A rotation of $\frac{\pi}{4}$ of our vector field $\mathbf{f}$



(a) $\mathbf{f}$ (b) $\mathbf{r} \bullet \mathbf{f}$

Figure 10: Vector field after a rotation of $\frac{\pi}{4}$

Our vector field is invariant with rotations.

Stabilisers III

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Stabilisers III

- stabiliser: transformation with an action that preserves a vector field
- symmetry : change of variables that preserves the equation form

$\rightarrow$ *transformation* $\mathbf{g}$ is a stabiliser $\implies \mathbf{g} \in Sym(\mathbf{f})$
$\rightarrow \mathbf{g_1} \circ \mathbf{g_2} \in Sym(\mathbf{f})$

Stabilisers IV

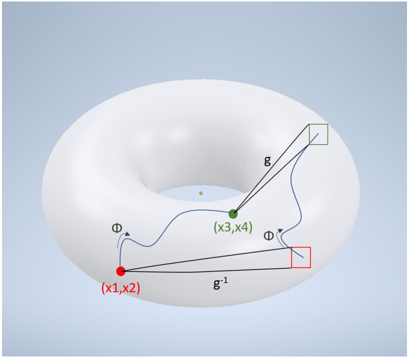


Figure 11: Example of stabiliser on the torus

Implementation

3 steps:

- finding a stabiliser according to the symmetries of our system
- calculate a reference trajectory using a guaranteed integration tool
- apply transformation with our particular parameters using the stabiliser

Our system follows the given equations: $\begin{cases} \dot{x}_1 = -x_1^3 - x_1 x_2^2 + x_1 - x_2 \\ \dot{x}_2 = -x_2^3 - x_1^2 x_2 + x_1 + x_2 \end{cases}$

This gives us the following vector field shown below

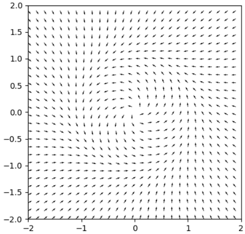


Figure 12: Example 1 vector field

Example 1: Circle with attractive point II

We calculate a reference trajectory with the initial condition $\mathbf{a_0} = \begin{pmatrix} 0 \\ \frac{1}{2} \end{pmatrix}$ over 10s

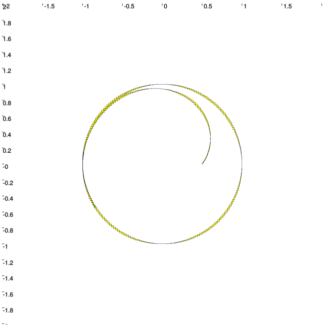


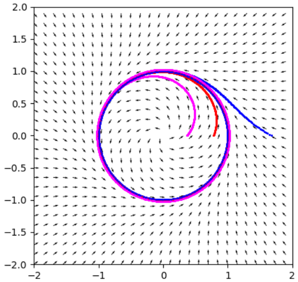
Figure 13: Example 1 Reference trajectory

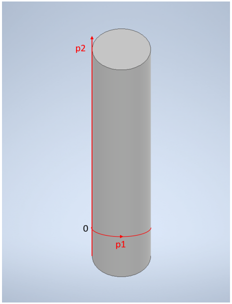
Example 1: Circle with attractive point III

Stabilisers satisfy the equation $\mathbf{g} \bullet \mathbf{f} = \mathbf{f}$

- Rotation : $g_1 : \mathbf{x} \rightarrow \mathbf{R}(p_1) * \mathbf{x}$
- Symmetry with respect to the unit circle
  $g_2 : \mathbf{x} \rightarrow \frac{1}{\sqrt{p_2 + ({x_1}^2 + {x_2}^2)(1 - p_2)}} * \mathbf{x}$

A group of stabilisers given by $\mathbf{g_p} = g_2 \circ g_1$ with $\mathbf{p} = \begin{pmatrix} p_1 \\ p_2 \end{pmatrix}$ exists.





Example 1: Circle with attractive point IV

For our example the transformation to go from the reference trajectory to the one with our initial condition is:

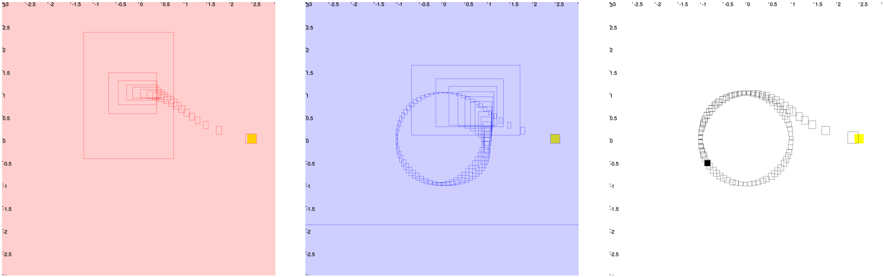
$$\mathbf{x}(t) = \mathbf{g}_{\mathbf{h}(\mathbf{x_0})} \circ \mathbf{a}(t)$$

$$\mathbf{x}(t) = \frac{\sqrt{3}\begin{pmatrix} x_1(0), -x_2(0) \\ x_2(0), x_1(0) \end{pmatrix}\begin{pmatrix} a_1(t) \\ a_2(t) \end{pmatrix}}{\sqrt{1 - n^2(\mathbf{x_0}) + n^2(\mathbf{a}(t))}(4n^2(\mathbf{x_0}) - 1)}$$

Example 1: Circle with attractive point V

We now want to compare the performances for a non degenerate box, for instance

$$[\mathbf{x_0}] = \begin{bmatrix} [2.4, 2.6] \\ [-0.1, 0.1] \end{bmatrix}$$



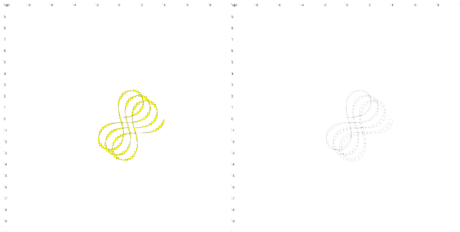
(c) CAPD (d) DynIbex (e) Lie Symmetries

Figure 14: Integration comparison

In our second example, the system follows the given equations:

$$\begin{cases} \dot{x_1} = cos(x_3) \\ \dot{x_2} = sin(x_3) \\ \dot{x_3} = sin(0.4 * x_4) \\ \dot{x_4} = 1 \end{cases}$$

We will use as initial condition for our reference the point $\mathbf{a0} = (0, 0, 0, 0)$

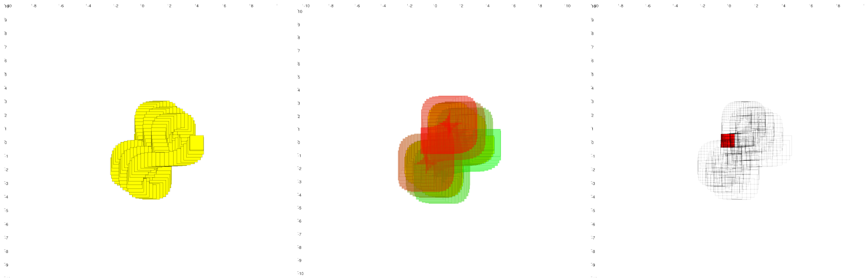


(a) CAPD (b) DynIbex

Figure 15: Reference trajectory

Case 1

$$\text{Initial box: } [\mathbf{x_0}] = \begin{bmatrix} [-0.5, 0.5] \\ [-0.5, 0.5] \\ [0, 0] \\ [0, 0] \end{bmatrix}$$



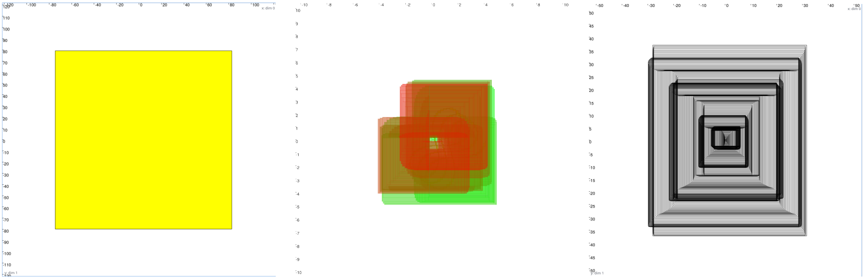
(a) CAPD (b) Lie Group (c) DynIbex

Figure 16: Example 2 Experiment 1 results

Example 2: trajectory without attractor III

Case 2

Initial box: $[\mathbf{x_0}] = \begin{bmatrix} [0,0] \\ [0,0] \\ [-\frac{\pi}{2}, \frac{\pi}{2}] \\ [0,0] \end{bmatrix}$



(a) CAPD (b) Lie Group (c) DynIbex

Figure 17: Example 2 Experiment 2 results

Limits

As powerful as it can be this method is still limited:

1. There may be no symmetries for a problem
2. In case the symmetries exist, it is not always easy to find the associated transformation

Conclusion

During this session we presented a novel approach for guaranteed integration using symmetries in Lie groups. This new method based on transformations from a reference trajectory allows us to save computational time and memory. It is robust when the problem present an attractor and is particularly able to face cases with uncertainties on multidimensional problems in case symmetries exist

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Thank you for your attention

Appendices

Definition

A diffeomorphism is an isomorphism from a smooth manifold to another. It is invertible, smooth and its inverse is also smooth

Example 1: Circle with attractive point TABLE

	CAPD	DynIbex	Lie Symmetries
computing time for reference	140ms	7310ms	
final box for reference curve	$\begin{bmatrix}[-0.849, -0.839]\\ [-0.544, -0.527]\end{bmatrix}$	$\begin{bmatrix}[-0.841, -0.837]\\ [-0.546, -0.54]\end{bmatrix}$	
computing time for larger box	116ms	16309ms	140ms
last step computed	1.856s	1.4241s	10s

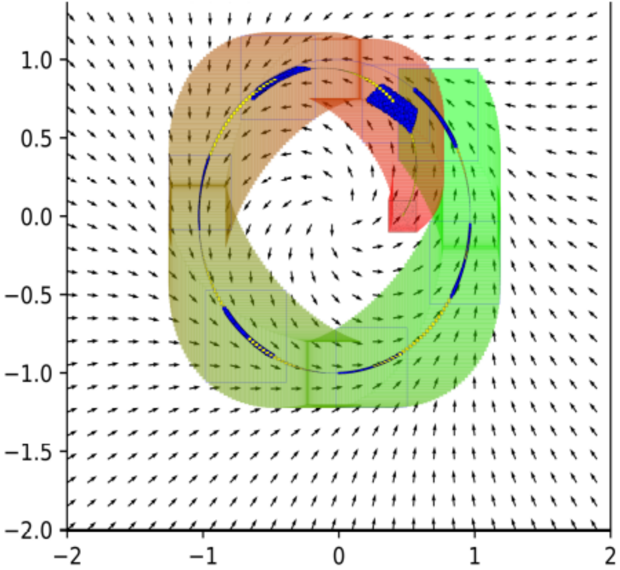
Table 1: Example 1 comparisons

Example 2: trajectory without attractor TABLE

	CAPD	DynIbex	Lie Symmetries
computing time for 1st experiment	40ms	9582ms	40ms
final box for 1st experiment	$\begin{bmatrix}[3.586, 4.601]\\ [-0.674, 0.397]\end{bmatrix}$	$\begin{bmatrix}[3.535, 4.655]\\ [-0.660, 0.453]\end{bmatrix}$	$\begin{bmatrix}[3.584, -4.602]\\ [-0.673, 0.397]\end{bmatrix}$
computing time 2nd experiment	37ms	11893ms	41ms
final box for 2nd experiment	$\begin{bmatrix}[-74.622, 82.717]\\ [-79.703, 79.202]\end{bmatrix}$	$\begin{bmatrix}[-28.153, 31.8]\\ [-37.046, 36.943]\end{bmatrix}$	$\begin{bmatrix}[-0.178, 4.283]\\ [-4.283, 4.102]\end{bmatrix}$

Table 2: Example 2 results

Tube with particles



Set Inversion